STATS 701 – Theory of Reinforcement Learning
Markov Decision Processes, part 2

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Outline

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1. Markov Decision Processes: Solution Methodologies
   - Value of a Policy by Iteration
   - Action-Value Function and the Bellman Optimality Equation
   - Value and Policy Iterations
   - Linear Programming Formulation

Value of a Policy by Iteration

Each policy $\pi \in \Pi_d$ results in a Markov process with state transition matrix A^π and reward function R_π.

Define the vector $\mathbf{v}_\pi$ by

$$\mathbf{v}_\pi = [\ V_\pi(x_1) \quad \ldots \quad V_\pi(x_n)\ ],$$

and the reward vector $\mathbf{r}_\pi$ by

$$\mathbf{r}_\pi = [\ R_\pi(x_1) \quad \ldots \quad R_\pi(x_n)\ ].$$

Then $\mathbf{v}_\pi$ satisfies the familiar relation

$$\mathbf{v}_\pi = \mathbf{r}_\pi + \gamma A^\pi \mathbf{v}_\pi.$$

This equation can be solved by iteration, as before.

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Acton-Value Function

Definition

The action-value function $Q : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is defined by

$$Q_\pi(x_i, u_k) := E_\pi\left[\sum_{t=0}^{\infty} \gamma^t R_\pi(X_t) | X_0 = x_i, U_0 = u_k\right].$$

Note: In the definition of $Q_\pi(x_i, u_k)$, at the first instant $t = 0$ we choose the action u_k as we wish, not necessarily as $u_k = \pi(x_i)$.

But for $t \geq 1$, we choose $U_t = \pi(X_t)$.

Q can be viewed as a real vector of dimension $|\mathcal{X}| \cdot |\mathcal{U}|$.

Recursive Relationshop of Action-Value Function

Theorem

The function Q satisfies the recursive relationship

$$Q_\pi(x_i, u_k) = R(x_i, u_k) + \gamma \sum_{j=1}^{n} a_{ij}^{u_k} Q_\pi(x_j, \pi(x_j)).$$

Proof in the notes.

Relationship Between Action-Value and Value Functions

Theorem

The functions V_π and Q_π are related via

$$V_\pi(x_i) = Q_\pi(x_i, \pi(x_i)).$$

In view of this theorem, the recursive equation for Q_π, namely

$$Q_\pi(x_i, u_k) = R(x_i, u_k) + \gamma \sum_{j=1}^{n} a_{ij}^{u_k} Q_\pi(x_j, \pi(x_j)).$$

can be rewritten as

$$Q_\pi(x_i, u_k) = R(x_i, u_k) + \gamma \sum_{j=1}^{n} a_{ij}^{u_k} V_\pi(x_j).$$

Optimal Value Function and Optimal Policy

Let $V^*(x_i)$ denote the maximum value of the discounted future reward, over all policies $\pi \in \Pi_d$:

$$V^*(x_i) := \max_{\pi \in \Pi_d} V_\pi(x_i).$$

Note that, though the set Π_d may be huge, it is nevertheless a finite set. Therefore the maximum above exists. Also define

$$\pi^* = \arg\max_{\pi \in \Pi_d} V_\pi(x_i).$$

Thus π^* is any policy such that $V_{\pi^*} = V^*$.

Bellman Optimality Equation

Theorem

Define $V^*(x_i)$ as above. Then $V^*(x_i)$ satisfies the reursive relationship

$$V^*(x_i) = \max_{u_k \in \mathcal{U}} \left[R(x_i, u_k) + \gamma \sum_{j \in [n]} a_{ij}^{u_k} V^*(x_j) \right].$$

This is known as the Bellman optimality equation. Note that it does not help us to find the function V^*; it just a characterization of V^*.

Rationalization of the Bellman Optimality Equation

Suppose that we have somehow determined the maximum possible value $V^*(1, x_j)$ at time $t = 1$ for each state $x_j \in \mathcal{X}$. Now at time $t = 0$, suppose the state $X_0 = x_i$. Then each action $u_k \in \mathcal{U}$ leads to the value

$$R(x_i, u_k) + \gamma \sum_{j \in [n]} a_{ij}^{u_k} V^*(1, x_j).$$

So the maximum value at time $t = 0$, state $X_0 = x_i$ is given by

$$V^*(0, x_i) = \max_{u_k \in \mathcal{U}} \left[R(x_i, u_k) + \gamma \sum_{j \in [n]} a_{ij}^{u_k} V^*(1, x_j) \right].$$

However, since both the MDP and policy are time-invariant, we must have that $V(1, x_i) = V(0, x_i)$ for each $x_i \in \mathcal{X}$.

Recursive Relationship for Optimal Action-Value Function

Theorem

Define

$$Q^*(x_i, u_k) = R(x_i, u_k) + \gamma \sum_{j=1}^{n} a_{ij}^{u_k} V^*(x_j).$$

Then $Q^*(\cdot, \cdot)$ *satisfies the relationship*

$$Q^*(x_i, u_k) = R(x_i, u_k) + \gamma \sum_{j=1}^{n} a_{ij}^{u_k} \max_{w_l \in \mathcal{U}} Q^*(x_j, w_l).$$

Note that the maximum w.r.t. $u_k \in \mathcal{U}$ is in a different place compared to the Bellman equation. This is crucial.

Advantage of Optimal Action-Value Function

Theorem

Once $Q^*(\cdot,\cdot)$ is determined, we have that

$$V^*(x_i) = \max_{u_k \in \mathcal{U}} Q^*(x_i, u_k),$$

$$\pi^*(x_i) = \arg\max_{u_k \in \mathcal{U}} Q^*(x_i, u_k),$$

Moreover, it is easier to "learn" $Q^*(\cdot,\cdot)$ than to "learn" $V^*(\cdot)$, as we shall see.

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Value Update Map

Define the optimal value vector $\mathbf{v}^*$ as

$$\mathbf{v}^* = [\ V^*(x_1) \quad \cdots \quad V^*(x_n)\ ].$$

Next, define a "value update" map $T : \mathbb{R}^n \to \mathbb{R}^n$, as follows:

$$(T\mathbf{v})_i := \max_{u \in \mathcal{U}} \left[R(x_i, u) + \gamma \sum_{j \in [n]} a_{ij}^u v_j \right].$$

One can think of $\mathbf{v}$ as the current guess for the vector $\mathbf{v}^*$, and of $T\mathbf{v}$ as an updated guess. The next theorem shows that this intuition is valid.

Value Iteration

Theorem

The map T is both monotone and a contraction. As a result, for all $\mathbf{v}_0 \in \mathbb{R}^n$, the sequence of iterations $\{T^k \mathbf{v}_0\}$ approaches $\mathbf{v}^*$ as $k \to \infty$.

Value Iteration Proof

Monotonicity is easy to prove, hence focus on contraction

$$
\begin{aligned}
&|(T\mathbf{w})_i - (T\mathbf{v})_i| \\
&= |\max_u [R(x_i, u) + \gamma \sum_j a^u_{ij} w_j] - \max_u [R(x_i, u) + \gamma \sum_j a^u_{ij} v_j]| \\
&\leq \max_u |\gamma \sum_j a^u_{ij} w_j - \gamma \sum_j a^u_{ij} v_j| \\
&= \gamma \max_u |(A^u w)_i - (A^u v)_i| \leq \gamma \max_u \|A^u \mathbf{w} - A^u \mathbf{v}\|_\infty \\
&\leq \gamma \max_u \|A^u\|_{\infty \to \infty} \|\mathbf{w} - \mathbf{v}\|_\infty \\
&\leq \gamma \|\mathbf{w} - \mathbf{v}\|_\infty
\end{aligned}
$$

Action-Value Iteration

Define $\mathbf{q} \in \mathbb{R}^{|\mathcal{X}|\times|\mathcal{U}|}$ as the vector $[Q(x_i, u_k), x_i \in \mathcal{X}, u_k \in \mathcal{U}]$. Define $F : \mathbb{R}^{|\mathcal{X}|\times|\mathcal{U}|} \to \mathbb{R}^{|\mathcal{X}|\times|\mathcal{U}|}$ by

$$(F\mathbf{q})(x_i, u_k) := R(x_i, u_k) + \gamma \sum_{j=1}^{n} a_{ij}^{u_k} \max_{w_l \in \mathcal{U}} Q(x_j, w_l).$$

Theorem

The map F is monotone and is a contraction. Therefore for all $\mathbf{q}_0 \in \mathbb{R}^{|\mathcal{X}|\times|\mathcal{U}|}$, the sequence of iterations $\{F^k(\mathbf{q}_0)\}$ converges to $\mathbf{q}^*$.

Determining Optimal Policy from Optimal Value Function

Determining the optimal policy π^* from the optimal value function is easy.

Theorem

Suppose the optimal value vector $\mathbf{v}^*$ is known, and define, for each $x_i \in \mathcal{X}$, the policy $\pi^* : \mathcal{X} \to \mathcal{U}$ via

$$\pi^*(x_i) = \arg\max_{u_k \in \mathcal{U}} \left[R(x_i, u) + \gamma \sum_{j \in [n]} a_{ij}^{u_k} V^*(x_j) \right].$$

But this requires knowledge of the optimal value function $\mathbf{v}^*$.

Is there another way? Yes, to update policy along with value iteration.

Policy Iteration

Set iteration counter $k = 0$ and choose some initial policy π_0. Then at iteration k,

- Compute the value vector $\mathbf{v}_{\pi_k}$ such that $\mathbf{v}_{\pi_k} = T_{\pi_k}\mathbf{v}_{\pi_k}$. Note that computing $\mathbf{v}_{\pi_k}$ by value iteration would require infinitely many applications of the map T_{π_k} to some arbitrary initial vector.
- Use this value vector $\mathbf{v}_{\pi_k}$ to compute an updated policy π_{k+1}, via

$$\pi_{k+1}(x_i) = \arg\max_{u_k \in \mathcal{U}} \left[R(x_i, u_k) + \gamma \sum_{j \in [n]} a_{ij}^{u_k} (\mathbf{v}_{\pi_k})_j \right].$$

Policy Iteration (Cont'd)

Note that the above equation implies that

$$T_{\pi_{k+1}} \mathbf{v}_{\pi_k} = T \mathbf{v}_{\pi_k} \geq T_{\pi_k} \mathbf{v}_{\pi_k} = \mathbf{v}_{\pi_k}$$

where the last eq. is because v_{π_k} is the value function of π_k

Since the map $T_{\pi_{k+1}}$ is monotone, we can keep applying it to get

$$\forall \ell \geq 1, \; T^{\ell}_{\pi_{k+1}} \mathbf{v}_{\pi_k} \geq \mathbf{v}_{\pi_k}$$

Note: LHS converges to $\mathbf{v}_{\pi_{k+1}}$ as $\ell \to \infty$

Convergence of Policy Iteration

Suppose policy iteration doesn't improve the policy, i.e. $\pi_{k+1} = \pi_k$ and the inequality $T\mathbf{v}_{\pi_k} \geq \mathbf{v}_{\pi_k}$ is equality. Then we have

$$T\mathbf{v}_{\pi_k} = \mathbf{v}_{\pi_k}$$

So $\mathbf{v}_{\pi_k}$ must be the optimal value function and the corresponding greedy policy $\pi_{k+1} = \pi_k$ must be the optimal policy.

Theorem

We have that

$$\mathbf{v}_{\pi_{k+1}} \geq \mathbf{v}_{\pi_k},$$

where the dominance is componentwise. Consequently, there exists a finite integer k_0 such that $\mathbf{v}_{\pi_k} = \mathbf{v}^*$ for all $k \geq k_0$.

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LP: Primal

Let $\mathbf{d}$ be any distribution over states

$$\min_{\mathbf{v}} \mathbf{d}^\top \mathbf{v}$$

$$\text{s.t. } \mathbf{v} \geq T\mathbf{v}$$

It is clear that $\mathbf{v}^\star$ is feasible and therefore the minimum is at most $\mathbf{d}^\top \mathbf{v}^\star$

Claim: the minimum above is equal to $\mathbf{d}^\top \mathbf{v}^\star$
Why? $\mathbf{v} \geq T\mathbf{v}$ implies $\mathbf{v} \geq T^\ell \mathbf{v} \Rightarrow \mathbf{v} \geq \mathbf{v}^\star \Rightarrow \mathbf{d}^\top \mathbf{v} \geq \mathbf{d}^\top \mathbf{v}^\star$

LP: equivalent form

$$\min_{\mathbf{V}} \sum_{x_i \in \mathcal{X}} d(x_i) V(x_i)$$

$$\text{s.t. } \forall x_i \in \mathcal{X}, \quad V(x_i) \geq \max_{u_k \in \mathcal{U}} \; R(x_i, u_k) + \gamma \sum_{x_j \in \mathcal{X}} a_{ij}^{u_k} V(x_j)$$

LP: equivalent form

$$\min_{\mathbf{v}} \sum_{x_i \in \mathcal{X}} d(x_i) V(x_i)$$

$$\text{s.t. } \forall x_i \in \mathcal{X}, u_k \in \mathcal{U}, \quad V(x_i) \geq R(x_i, u_k) + \gamma \sum_{x_j \in \mathcal{X}} a_{ij}^{u_k} V(x_j)$$

This LP has n unconstrained variables and mn inequality constraints
Dual LP will have mn non-negative variables and n equality constraints

LP Duality

The linear program

$$\min_{\mathbf{x}} \ \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t. } A\mathbf{x} \geq \mathbf{b}$$

has the dual

$$\max_{\mathbf{y} \geq 0} \ \mathbf{b}^\top \mathbf{y}$$

$$\text{s.t. } A^\top \mathbf{y} = \mathbf{c}$$

LP: Dual

$$\max_{\boldsymbol{\mu} \geq 0} \sum_{x_i \in \mathcal{X}, u_k \in \mathcal{U}} \mu(x_i, u_k) R(x_i, u_k)$$

$$\text{s.t. } \forall x_i \in \mathcal{X}, \sum_{u_k \in \mathcal{U}} \mu(x_i, u_k) = d(x_i) + \gamma \sum_{x_j \in \mathcal{X}} \sum_{u_k \in \mathcal{U}} A_{ij}^{u_k} \mu(x_j, u_k)$$

Interpretation of $\boldsymbol{\mu}$: discounted state-action visitation frequencies

$$\mu(x_j, u_k) = \sum_{t=0}^{\infty} \gamma^t P(X_t = x_j, U_t = u_k)$$

Solution directly encodes optimal policy: $\pi^\star(x_j) = \arg\max_{u_k} \boldsymbol{\mu}^\star(x_j, u_k)$